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SET SYSTEMS WITH FEW DISJOINT PAIRS

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Let $X=\{1,\dots,n\}$, and let $\mathcal{A}$ be a family of subsets of X. Given the size of $\mathcal{A}$, at least how many pairs of elements of $\mathcal{A}$ must be disjoint? In this paper we give a lower bound for the number of disjoint pairs in $\mathcal{A}$. The bound we obtain is essentially best possible. In particular, we give a new proof of a result of Frankl and of Ahlswede, that if $\mathcal{A}$ satisfies $|\mathcal{A}|=\left|X^{(\geq r)}\right|$ then $\mathcal{A}$ contains at least as many disjoint pairs as $X^{(\geq r)}$.

The situation is rather different if we restrict our attention to $\mathcal{A}\subset X^{(r)}$: then we are asking for the minimum number of edges spanned by a subset of the Kneser graph of given size. We make a conjecture on this lower bound, and disprove a related conjecture of Poljak and Tuza on the largest bipartite subgraph of the Kneser graph.

0. Introduction

Let $X=\{1,\dots,n\}$, and let $\mathcal{A}$ be a family of subsets of X. Given the size of $\mathcal{A}$, how few disjoint pairs can $\mathcal{A}$ contain?

To make this question precise, let us introduce a little notation. Define a graph on $\mathcal{P}(X)$, the power set of X, as follows: given distinct $A,B\in\mathcal{P}(X)$, we join A to B if $A\cap B=\emptyset$ (note that this a simple graph, as there is no loop at $\emptyset$). Given a set system $\mathcal{A}\subset\mathcal{P}(X)$, write $D(\mathcal{A})$ for the number of edges spanned by $\mathcal{A}$. Then we wish to minimize $D(\mathcal{A})$ for a given value of $|\mathcal{A}|$. One could view $D(\mathcal{A})$ as a measure of how non-intersecting $\mathcal{A}$ is.

For $|\mathcal{A}|\leq 2^{n-1}$, we may take $\mathcal{A}$ to be an intersecting family (ie. $A\cap B\neq\emptyset$ whenever $A,B\in\mathcal{A}$), so that $D(\mathcal{A})=0$. We remark that there are then many

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extremal systems: indeed, it is easy to see that every maximal intersecting family has size 2^{n-1}. (See [4, Ch. 4] or [10] for general background on intersecting families.)

However, the question becomes non-trivial for $|\mathcal{A}| > 2^{n-1}$. There are now not so many extremal systems. In contrast to the case $|\mathcal{A}| \leq 2^{n-1}$, one finds that one cannot just choose the members of $\mathcal{A}$ 'greedily'. For example, with $|\mathcal{A}| = 2^{n-1}+2$ it is not the case that every intersecting system extends to an extremal system (the reader might care to check that the intersecting family $\{A \subset X : 1 \in A\}$ is a suitable example). A little experiment suggests that the best sets (those $\mathcal{A}$ with smallest $D(\mathcal{A})$ for their size) are those of the form $X^{(\geq r)} = \{A \subset X : |A| \geq r\}$.

Frankl [9] and, independently, Ahlswede [1] proved that sets of the form $X^{(\geq r)}$ are indeed best. More precisely, they showed that if $\mathcal{A} \subset \mathcal{P}(X)$ satisfies $|\mathcal{A}| = \left|X^{(\geq r)}\right|$ then $D(\mathcal{A}) \geq D(X^{(\geq r)})$. In Section 1, we shall give a new proof of this result. We also give a fairly good lower bound on $D(\mathcal{A})$ for every other value of $|\mathcal{A}|$.

Let us point out one feature that does *not* follow from our methods, but is proved by Frankl [9] and Ahlswede [1]. Each of them actually shows that, given a system $\mathcal{A}$ with $\left|X^{(\geq r+1)}\right| < |\mathcal{A}| < \left|X^{(\geq r)}\right|$, there is a system $\mathcal{B}$ with $X^{(\geq r+1)} \subset \mathcal{B} \subset X^{(\geq r)}$ satisfying $|\mathcal{B}| = |\mathcal{A}|$ and $D(\mathcal{B}) \leq D(\mathcal{A})$.

Given the result above, one might guess that there is an ordering on $\mathcal{P}(X)$, with $X^{(\geq r)}$ an initial segment for every r, such that initial segments of this ordering minimize $D(\mathcal{A})$ for given $|\mathcal{A}|$. Rather interestingly, this is *not* the case. For example, suppose that $X^{(\geq r+1)} \subset \mathcal{A} \subset X^{(\geq r)}$ $(r < n/2)$, with $|\mathcal{A}| = \left|X^{(\geq r+1)}\right| + \binom{n-1}{r-1}$. Then, to minimize $D(\mathcal{A})$, the Erdős–Ko–Rado theorem [8], on the largest intersecting family in $X^{(r)} = \{A \subset X : |A| = r\}$, tells us that we should take $\mathcal{A} = X^{(\geq r+1)} \cup \left\{A \in X^{(r)} : i \in A\right\}$, for any fixed $i \in X$, and that these are the unique extremal systems. However, now suppose instead that we have $X^{(\geq r+1)} \subset \mathcal{A} \subset X^{(\geq r)}$ with $|\mathcal{A}| = \left|X^{(\geq r+1)}\right| + \binom{n-1}{r}$. In this case, using the fact that the Kneser graph $K(n,r)$ (the graph on $X^{(r)}$ in which A is joined to B if $A \cap B = \emptyset$) is regular, it is easy to see that the same Erdős–Ko–Rado theorem tells us that the unique extremal systems are those of the form $\mathcal{A} = X^{(\geq r+1)} \cup \left\{A \in X^{(r)} : i \notin A\right\}$, $i \in X$. And of course no set of the form $\{A \subset X : i \notin A\}$ contains a set of the form $\{A \subset X : j \in A\}$, so that the extremal sets cannot form a nested family.

The fact that there is no ordering suggests that, in order to prove our desired result, we almost certainly cannot just rely on compression operators, as often used in solving extremal problems (see eg. [5, 10, 15]). It turns out that the key tool to use is the notion of 'fractional set systems': roughly speaking, the idea is that, to compare our system $\mathcal{A}$ with the desired extremal system $X^{(\geq r)}$, we need to create a smooth passage from $\mathcal{A}$ to $X^{(\geq r)}$. To bypass the lack of an ordering, we need to pass via some 'non-existent' set systems. As we shall see, once we have defined the notion of a fractional set system, and once we have decided what is meant by the number of disjoint pairs in a fractional set system, it will be fairly easy to prove our result. It is important to point out that fractional systems are being used as a tool to prove results about genuine (non-fractional) systems.

Our result also extends to t-disjoint pairs, in other words pairs A, B of sets with $|A \cap B| < t$. We show that if $\mathcal{A}$ is a set system with $|\mathcal{A}| = \left|X^{(\geq r)}\right|$ then $\mathcal{A}$ contains at least as many t-disjoint pairs as $X^{(\geq r)}$. This is particularly interesting in light of the example given by Frankl [9] that shows that, for t-intersecting, it is no longer true that it is always best to take a system $\mathcal{A}$ with $X^{(\geq r+1)} \subset \mathcal{A} \subset X^{(\geq r)}$. In the special case when $n+t = 2r$, our proof reduces to a particularly simple proof of the theorem of Katona [12] that, for $n+t$ even, the largest t-intersecting family in $\mathcal{P}(X)$ is $X^{(\geq (n+t)/2)}$.

In Section 2, we turn our attention to the case $\mathcal{A} \subset X^{(r)}$, in other words to the question of the minimum number of edges spanned by a subset of the Kneser graph $K(n,r)$ of given size. As we have seen, the extremal sets do not form a nested family. We make a conjecture on the minimum value of $D(\mathcal{A})$ in terms of $|\mathcal{A}|$.

Since the Kneser graph is regular, the problem of minimizing the number of induced edges of $\mathcal{A}$ is the same as that of maximizing the number of edges between $\mathcal{A}$ and $X^{(r)} \setminus \mathcal{A}$. Poljak and Tuza [13] asked for the maximum number of edges from $\mathcal{A}$ to $X^{(r)} \setminus \mathcal{A}$ over *all* sets $\mathcal{A} \subset X^{(r)}$ (not just those of a given size). In other words, they were asking for the largest size of a bipartite subgraph of the Kneser graph. They conjectured that the extremal value is obtained by taking one of the sets $\mathcal{A}_k = \left\{A \in X^{(r)} : \{1,\dots,k\} \subset A\right\}$, $k = 1,\dots,r-1$. They proved this for $r = 2$ and also for $r \geq n/(4.3+o(1))$ (although, as we shall remark later, the proof of the latter assertion is not correct). Although the Poljak–Tuza conjecture is consistent with the Erdős–Ko–Rado estimates mentioned earlier, it is not consistent with our conjecture on the minimum value of $D(\mathcal{A})$ for given $|\mathcal{A}|$. In fact, it turns out that the conjecture of Poljak and Tuza is not correct: in Section 2 we give a counterexample. We also make a modified conjecture on the largest bipartite subgraph of $K(n,r)$.

Let us remark that the 'opposite' of our main question, namely the problem of *maximizing* the number of disjoint pairs in a set system of given size, has also been studied; see Alon and Frankl [2] for several important results on this and related topics.

1. Minimizing the number of disjoint pairs

Our aim in this section is to show that if $\mathcal{A}$ is a set system on X with $|\mathcal{A}| \geq \left|X^{(\geq r)}\right|$ then $D(\mathcal{A}) \geq D(X^{(\geq r)})$, and to obtain a good lower bound on $D(\mathcal{A})$ for other values of $|\mathcal{A}|$.

One would like to make use of compression operators to prove this result. In other words, one would like to replace $\mathcal{A}$ by a set system $\mathcal{A}'$, with $|\mathcal{A}'| = |\mathcal{A}|$ and $D(\mathcal{A}') \leq D(\mathcal{A})$, such that $\mathcal{A}'$ 'looks more like' $X^{(\geq r)}$ than $\mathcal{A}$ did. The hope would be that one can repeat this, obtaining $\mathcal{A}''$, $\mathcal{A}'''$, and so on, until eventually we obtain either precisely $X^{(\geq r)}$ or else a system so close to $X^{(\geq r)}$ that one can verify directly that it contains at least as many disjoint pairs as $X^{(\geq r)}$ does. Unfortunately, for compressions to be manageable (in our case, to prove that $D(\mathcal{A}') \leq D(\mathcal{A})$), one invariably needs that there is an ordering whose initial segments are all extremal – for examples of this, see eg. [5, 10, 15].

To get around this problem, we need to introduce some 'non-existent' set systems, just to allow us to compress $\mathcal{A}$ and hope to arrive at some ordering. A *fractional set system*, or simply a *system*, on X is a function $f : \mathcal{P}(X) \to [0,1] \subset \mathbb{R}$. Fractional systems f with $f(\mathcal{P}(X)) \subset \{0,1\}$ correspond to (non-fractional) set systems, putting $\mathcal{A} = f^{-1}(0)$. Fractional set systems first appeared in [6], where they were introduced in the 'weighted cube' because there might be too few set systems of a given weight to allow sensible compressions. Here, the aim is rather different: they are being used to make up for a lack of an ordering in the cube.

As in [6], the *weight* of a system f is

$$w(f) = \sum_{A \in \mathcal{P}(X)} f(A).$$

A *fractional Hamming ball*, or simply a *ball*, is a fractional system f of the form

$$f(A) = \begin{cases} 1 & \text{if } |A| > r \\ \alpha & \text{if } |A| = r \\ 0 & \text{if } |A| < r \end{cases}$$

for some $0 \leq r \leq n$ and $0 \leq \alpha \leq 1$. For $0 \leq w \leq 2^n$ we write b^w for the unique ball of weight w.

Our task now is to define $D(f)$, the 'number of disjoint pairs in f', in such a way that all our compressions will work smoothly, and in particular so that it is plausible that, among all systems f of given weight, the one with minimal $D(f)$ is the ball of that weight. It turns out that a suitable definition to take is

$$D(f) = \sum_{AB\in E} (f(A)+f(B)-1)^+,$$

where x^+ denotes $\max(x,0)$ (and the sum is over all edges of our graph on $\mathcal{P}(X)$). Note that if $F(\mathcal{P}(X))\subset\{0,1\}$, so that f corresponds to a set system $\mathcal{A}$, then we do have $D(f)=D(\mathcal{A})$.

Let us introduce a quantity that is slightly easier to work with than $D(\mathcal{A})$. For set systems $\mathcal{A},\mathcal{B}\subset\mathcal{P}(X)$, define the cross-intersection function $D(\mathcal{A},\mathcal{B}) = |\{(A,B)\in\mathcal{A}\times\mathcal{B}\colon A\in\mathcal{A},\, B\in\mathcal{B},\, A\cap B=\emptyset\}|$. Thus $D(\mathcal{A}) = \frac{1}{2}D(\mathcal{A},\mathcal{A})$, unless $\emptyset\in\mathcal{A}$, in which case $D(\mathcal{A}) = \frac{1}{2}(D(\mathcal{A},\mathcal{A})-1)$. Our aim is in fact to prove that if $\mathcal{A},\mathcal{B}\subset\mathcal{P}(X)$ with $|\mathcal{A}|\ge\left|X^{(\ge r)}\right|$ and $|\mathcal{B}|\ge\left|X^{(\ge s)}\right|$ then $D(\mathcal{A},\mathcal{B})\ge D(X^{(\ge r)},X^{(\ge s)})$.

Similarly, for systems f and g on X we define

$$D(f,g) = \sum_{(A,B)\in\mathcal{A}\times\mathcal{B},\ A\cap B=\emptyset} (f(A)+g(B)-1)^+.$$

We are now ready to define the compression operators we shall be using. They will map set systems into fractional set systems but, since we shall be iterating them, we must define them for every fractional set system.

Let f be a fractional system on X, and let $1\le i\le n$. The *i-sections* of f are the systems f_{i+} and f_{i-} on the ground set $X-\{i\}$ defined by

$$f_{i+}(A) = f(A\cup i),$$

$$f_{i-}(A) = f(A),$$

for $A\in\mathcal{P}(X-i)$. Thus f is determined by its i-sections, and $w(f)=w(f_{i+})+w(f_{i-})$. The *i-compression* of f is the system $C_i(f)$ on $X-\{i\}$ whose sections are balls of the same weights as f_{i+} and f_{i-}:

$$C_i(f)_{i+} = b^{w(f_{i+})},$$

$$C_i(f)_{i-} = b^{w(f_{i-})},$$

where of course the balls are systems on $X-\{i\}$. Thus certainly $w(C_i(f))=w(f)$. We say that f is *i-compressed* if $C_i(f)=f$.

We are now ready for our main result.

Theorem 1. *Let f and g be systems on X, and let b and c be the balls with $w(b)=w(f)$ and $w(c)=w(g)$. Then $D(b,c)\le D(f,g)$.*

Proof. We proceed by induction on n. For $n=1$ the result is trivial: we just need to observe that to minimize the quantity

$$(f(\emptyset)+g(\emptyset)-1)^+ + (f(\emptyset)+g(1)-1)^+ + (f(1)+g(\emptyset)-1)^+$$

we should give $f(1)$ and $g(1)$ as much weight as possible. We therefore turn to the induction step.

We claim first that, for any systems f and g on X, and any $1\le i\le n$, we have $D(C_i(f),C_i(g))\le D(f,g)$. Indeed, writing f' for $C_i(f)$ and g' for $C_i(g)$, and suppressing the i, we have

$$D(f,g) = D(f_-,g_-) + D(f_-,g_+) + D(f_+,g_-)$$

and

$$D(f',g') = D(f'_-,g'_-) + D(f'_-,g'_+) + D(f'_+,g'_-).$$

However, by the induction hypothesis we have $D(f_-,g_-)\ge D(f'_-,g'_-)$, and similarly for the other two terms. Hence $D(f,g)\ge D(f',g')$, as required.

We now wish to show that there are systems $\tilde f$ and $\tilde g$ on X, with $w(\tilde f)=w(f)$, $w(\tilde g)=w(g)$ and $D(\tilde f,\tilde g)\le D(f,g)$, such that $\tilde f$ and $\tilde g$ are i-compressed for all i. This will be accomplished by a simple compactness argument.

Let G be the set of those ordered pairs (u,v) of fractional systems that satisfy $w(u)=w(f)$, $w(v)=w(g)$ and $D(u,v)\le D(f,g)$. Then G is a (non-empty) compact subset of the product space $[0,1]^{\mathcal{P}(X)}\times[0,1]^{\mathcal{P}(X)}$. Furthermore, from the above we know that if $(u,v)\in G$ then $(C_i(u),C_i(v))\in G$ for all i.

For $0\le r\le n$, and any system u, write $w^{(r)}(u)$ for $\sum_{A\in X^{(r)}} u(A)$. Let $H\subset G$ be the subset of G obtained by maximizing successively $w^{(n)}(u)$, $w^{(n)}(v)$, $w^{(n-1)}(u)$, $w^{(n-1)}(v),\dots,w^{(1)}(u),w^{(1)}(v)$ on G. Thus H is itself compact. By the definition of an i-compression, $(u,v)\in H$ implies $(C_i(u),C_i(v))\in H$ for all i. Also, by the definition of H it is clear that if $(u,v)\in H$ then for each i-section h of u or v there is an r with $0\le r\le n-1$ such that $h(A)=1$ whenever $|A|>r$ and $h(A)=0$ whenever $|A|<r$.

Let us now choose $(u,v)\in H$ to maximize $w(u^{1/2})+w(v^{1/2})$. We claim that u and v are i-compressed for every i. Indeed, if for some i we have that u or v is not i-compressed then, by the Cauchy-Schwarz inequality, we would have

$$w(C_i(u)^{1/2}) + w(C_i(v)^{1/2}) > w(u^{1/2}) + w(v^{1/2}).$$

We have thus obtained systems $\tilde f$ and $\tilde g$ as desired.

The proof of our theorem is now easy to complete. Let us deal first with the case $n=2$. If $\tilde f$ is not a ball then it must satisfy $\tilde f(\emptyset)=0$ and

$\tilde{f}(\{1,2\})=1$, and similarly for $\tilde{g}$. In all cases it is now routine to verify that $D(b,c) \le D(\tilde{f},\tilde{g})$. The most complicated case is when neither $\tilde{f}$ nor $\tilde{g}$ is a ball, and then we have $D(\tilde{f},\tilde{g})=(\tilde{f}(1)+\tilde{g}(2)-1)^+ + (\tilde{f}(2)+\tilde{g}(1)-1)^+$, which is clearly at least $(b(1)+c(2)-1)^+ + ((b(2)+c(1)-1)^+$.

We now turn to the case $n\ge 3$. We know that, for every i, every i section of $\tilde{f}$ is a ball, and similarly for $\tilde{g}$. However, since $n\ge 3$, a moment's thought shows that this implies that $\tilde{f}$ and $\tilde{g}$ are themselves balls. Thus $\tilde{f}=b$ and $\tilde{g}=c$. ∎

This gives the following result for (non-fractional) set systems. As stated earlier, the 'most interesting' case, namely $|\mathcal{A}|=\left|X^{(\ge r)}\right|$, was first proved by Frankl [9] and Ahlswede [1].

Corollary 2. *Let $\mathcal{A},\mathcal{B}\subset\mathcal{P}(X)$ with $|\mathcal{A}|=\left|X^{(\ge r)}\right|$ and $|\mathcal{B}|=\left|X^{(\ge s)}\right|$. Then $D(\mathcal{A},\mathcal{B})\ge D(X^{(\ge r)},X^{(\ge s)})$. In particular, if $\mathcal{A}\subset\mathcal{P}(X)$ satisfies $|\mathcal{A}|=\left|X^{(\ge r)}\right|$ then $D(\mathcal{A}) \ge D(X^{(\ge r)}) = \frac{1}{2}\sum_{s=r}^{n}\binom{n}{s}\sum_{t=r}^{n-s}\binom{n-s}{t}$.* ∎

Let us remark that our methods apply identically when we consider not disjoint pairs but t-disjoint pairs, that is pairs A,B with $|A\cap B|<t$. Indeed, for a set system $\mathcal{A}$, and $t=1,2,\ldots$, let us define $D_t(\mathcal{A})$ to be the number of unordered pairs A,B from $\mathcal{A}$ with $|A\cap B|<t$. More generally, for systems f and g, and $t=1,2,\ldots$, define

$$D_t(f,g) = \sum_{(A,B)\in\mathcal{A}\times\mathcal{B},\ |A\cap B|<t} (f(A)+g(B)-1)^+.$$

Theorem 3. *Let f and g be systems on X, and let b and c be the balls with $w(b)=w(f)$ and $w(c)=w(g)$. Then for every $t=1,2,\ldots$ we have $D_t(b,c)\le D_t(f,g)$.*

Proof. The proof is just as for Theorem 1. The only complication for $t\ge 2$ is that the equation

$$D(f,g) = D(f_-,g_-) + D(f_-,g_+) + D(f_+,g_-)$$

is replaced by

$$D_t(f,g) = D_t(f_-,g_-) + D_t(f_-,g_+) + D_t(f_+,g_-) + D_{t-1}(f_+,g_+),$$

so that the induction on n is for all t at once. ∎

As before, as a particular case we have the following result for non-fractional set systems.

Corollary 4. *Let t be a positive integer, and let $\mathcal{A} \subset \mathcal{P}(X)$ satisfy $|\mathcal{A}| = \left|X^{(\geq r)}\right|$, some $r = 0, \ldots, n$. Then $D_t(\mathcal{A}) \geq D_t(X^{(\geq r)})$.* ∎

We wish to stress that here the methods of Frankl and Ahlswede do not seem to help. Indeed, as pointed out in Frankl [9], it is not always true that, given a system $\mathcal{A}$ with $\left|X^{(\geq r+1)}\right| < |\mathcal{A}| < \left|X^{(\geq r)}\right|$, there is a system $\mathcal{B}$ with $X^{(\geq r+1)} \subset \mathcal{B} \subset X^{(\geq r)}$ satisfying $|\mathcal{B}| = |\mathcal{A}|$ and $D_t(\mathcal{B}) \leq D_t(\mathcal{A})$.

It is perhaps worth pointing out that the method of proof of Corollary 4 gives a conceptually very clear proof of Katona's theorem [12] that, for $n+t$ even, the largest t-intersecting family in $\mathcal{P}(X)$ is $X^{(\geq (n+t)/2)}$.

Another way to generalize Theorem 1 would be to consider not pairs of disjoint sets but larger families: for example, one might count the number of pairwise disjoint triples $A, B, C \in \mathcal{A}$. It is easy to see that the proof of Theorem 1 extends to give the following result.

Theorem 5. *Let $\mathcal{A} \subset \mathcal{P}(X)$ with $|\mathcal{A}| = \left|X^{(\geq r)}\right|$. Then, for every k, the number of k-tuples from $\mathcal{A}$ that are pairwise disjoint is at least the number of such k-tuples from $X^{(\geq r)}$.* ∎

2. Induced edges in the Kneser graph

In this section we turn our attention to the situation when we restrict $\mathcal{A}$ to being a family of r-sets, for some fixed r. Which set systems now have the smallest number of disjoint pairs? To put it another way, which subsets of the Kneser graph $K(n,r)$ span the least edges? We assume throughout that $r < \frac{n}{2}$, as the result is trivial if $r \geq \frac{n}{2}$.

As we remarked earlier, the Erdős–Ko–Rado theorem tells us that if $|\mathcal{A}| = \binom{n-1}{r-1}$ then we should take $\mathcal{A}$ to be a set of the form $\left\{A \in X^{(r)} : i \in A\right\}$, while if $|\mathcal{A}| = \binom{n-1}{r}$ then we should take a set of the form $\left\{A \in X^{(r)} : i \notin A\right\}$. While this of course rules out any ordering, it does have some suggestive features. It suggests that for small sizes we should use initial segents of the lex ordering, while for large sizes we should use initial segments of the colex ordering. (See [4, Ch. 5] for general background on lex and colex.)

To put it another way, if $|\mathcal{A}|$ is small then we should ensure that small elements of the ground set belong to our sets, while if $|\mathcal{A}|$ is large then we should ensure that large elements of the ground set do not belong to our sets. What should we do in between? A little thought suggests the following way of 'moving along' from lex to colex. For $1 \le l \le r$ and $1 \le k \le n$, define

$$\mathcal{A}_{k,l} = \Big\{ A \in X^{(r)} : \ |A \cap \{1,\dots,k\}| \ge l \Big\}.$$

We call a set of the form $\mathcal{A}_{k,l}$, some k, an *l-ball*. Thus the 1-balls are initial segments of lex, while the r-balls are initial segments of colex.

We very much believe that, for each given size, one of these systems is the right one to take. To be a little more precise, let us define fractional balls in this generalised sense in the same way: for $1 \le l \le r$, a *fractional l-ball* is a fractional system f such that for every r-set A we have, writing $A = \{a_1, a_2, \dots, a_r\}$, where $a_1 < \dots < a_r$,

$$f(A) = \begin{cases} 1 & \text{if } a_l < k \\ \alpha & \text{if } a_l = k \\ 0 & \text{if } a_l > k \end{cases}$$

for some $0 \le k \le n$ and $0 \le \alpha \le 1$ (and of course $f(A) = 0$ if A is not an r-set). Thus the various fractional l-balls provide a smooth passage between the sets $\mathcal{A}_{k,l}$, $k = 1, \dots, n$.

Conjecture 6. Let $\mathcal{A} \subset X^{(r)}$. Then, for some $1 \le l \le r$, the fractional l-ball f of the same weight as $\mathcal{A}$ satisfies $D(f) \le D(\mathcal{A})$. ∎

Since the Kneser graph is regular, one could view the question above as asking for the most edges from a family of r-sets $\mathcal{A}$ of given size to its complement. In this formulation, one can drop the restriction on the size of $\mathcal{A}$, and just ask for the maximum number of edges between a subset of $K(n,r)$ and its complement. Equivalently, we are asking for the largest bipartite subgraph of $K(n,r)$. In this form, the question is a 'max cut' question – see Poljak and Tuza [14], Alon and Halperin [3] and Bollobás and Scott [7] for some sharp results for general graphs.

This question about $K(n,r)$ was studied by Poljak and Tuza [13]. They conjectured that the best set to take is always a set $\mathcal{A}_{k,1}$, for some $1 \le k \le n-1$. In other words, they conjectured that some 1-ball is always best. They proved this for the case $r = 2$ (the case of graphs), and also for $r \ge n/(4.3 + o(1))$. We remark that their proof of this latter result does not seem to be correct – the main lemma used, namely [13, Theorem 1], concerning 'sliced' colourings, is not correct.

Now, given our remarks above about the passage between lex and colex, there is no real reason to suppose that the largest bipartite subgraph should always be from a 1-ball to its complement (its complement being isomorphic to an r-ball, of course). Indeed, if we go up to $r = 3$ then there is an l that is not 1 or r, namely $l=2$. By symmetry, one would expect a 2-ball to have the best chance of winning when it has size $\frac{1}{2}\binom{n}{3}$. And this does happen at $n=14$. Indeed, the largest bipartite subgraph coming from a 1-ball has size 17985 (corresponding to the 1-ball $\mathcal{A}_{3,1}$), whereas the 2-ball $\mathcal{A}_{7,2}$ yields a bipartite subgraph of size 18130. Thus this is a counterexample to the conjecture of Poljak and Tuza.

We remark that it is not the case that the largest bipartite subgraph always comes from one of the two 'extremes' of 1-balls (equivalently, r-balls) or $\lfloor\frac{r+1}{2}\rfloor$-balls. For example, for $r=5$ and $n=23$, the largest bipartite graphs coming from 1-balls or 3-balls have sizes 86279760 and 85591044 respectively, whereas that coming from the 2-ball $\mathcal{A}_{7,2}$ has size 86545368.

In view of these examples, and Conjecture 6, we would propose the following modification to Poljak and Tuza's conjecture.

Conjecture 7. The largest bipartite subgraph of $K(n,r)$ is that with bipartition $\mathcal{A}_{k,l}$ and its complement, for some $1\le l\le r-1$ and some $1\le k\le n-1$. ∎

It is easy to check that Conjecture 7 would follow if we knew that Conjecture 6 was correct.

Fascinatingly, the question of which l one should take seems extremely delicate. For example, for $r=3$, one might guess that, since one has to wait until $n=14$ to have a 2-ball beating the best 1-ball, 2-balls should be best when n is large. But this is not the case. Indeed, one can check that the largest bipartite subgraph coming from a 1-ball $\mathcal{A}_{k,1}$ occurs when k is slightly below $n(1-2^{-1/3})$, with number of edges $\frac{1}{144}n^6-(\frac{1}{6}-\frac{1}{16}2^{1/3})n^5+O(n^4)$. The best 2-ball $\mathcal{A}_{k,2}$ occurs when k is $\lfloor\frac{n}{2}\rfloor$, with number of edges $\frac{1}{144}n^6-\frac{17}{192}n^5+O(n^4)$. So the largest bipartite subgraph comes from a 1-ball if n is large.

Of course, statements about r fixed and n large are not very exciting, since if n is much larger than r then the Kneser graph is very close to the complete graph, so that any bipartition into roughly equal classes will span about one half of the edges. Nevertheless, it is amusing to note that in fact, for $r=3$, the above example of $n=14$ is the unique value of n for which a 1-ball is not best or equal best.

Finally, let us point out that for the analogous question for the whole 'Kneser cube', in other words our usual graph on $\mathcal{P}(X)$, there is a rather simple answer: the largest bipartite subgraph of the Kneser cube has just over two-thirds of the edges.

Proposition 8. *A set system $\mathcal{A}\subset\mathcal{P}(X)$ has at most as many disjoint pairs (A,B) with $A\in\mathcal{A}$, $B\notin\mathcal{A}$ as the set system $\mathcal{A}_1=\{A\subset X:1\in A\}$.*

Proof. The edge from X to $\emptyset$ certainly goes between $\mathcal{A}_1$ and its complement. For the other edges, we proceed as follows.

Each partition of X into three disjoint sets A,B,C (apart from $X,\emptyset,\emptyset$) corresponds to three edges, namely AB, BC and CA. How many of these three edges go between $\mathcal{A}$ and its complement? Obviously, two, if some of A,B,C belong to $\mathcal{A}$ and some do not, and zero otherwise. Since the set system $\mathcal{A}_1$ has the property that in any partition of X into three sets we do not have all in $\mathcal{A}_1$ and we do not have none in $\mathcal{A}_1$, we are done. ∎

We remark that there are many other extremal examples. For example, if n is not a multiple of 3, we may take $\mathcal{A}=X^{(>n/3)}$.

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